A Model to Estimate the Dynamic Inverse for the Puerto Rican Economy

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Abstract

Due to the lack of data, in particular those related to each industrial sector's capital requirements, it has not been possible to develop dynamic models for the Puerto Rican economy. In this essay, Angel Ruíz presents a dynamic input-output model based on Leontief's dynamic inverse. About the dynamic inverse Professor PN Mathur has said: "A most important theoretical and empirical advance is Wassily Leontief's conceptualization and implementation of Dynamic Inverse. That gives us the direct and indirect input requirements generated by the final demand of the year zero for all previous individual years... if delivery to final demand at some specified future date is required, the method determines the 'productive advances' necessary today, and at every intervening point of time.... This gives measurable content to the notions of 'productive advances' of Quesnay, 'expanded reproduction' of Marx, and 'roundabout production' of Bohm-Bawerk."

I. Dynamic Input-Output Models

Introduction

A prerequisite for understanding Leontief's dynamic model is knowledge of' its static version. The static model basically answers the question as to which levels of output should be -produced by each of the e-industries in the economic system in order to satisfy the total demand for the products of these industries. The static model can be expressed and used in two versions, depending on the purposes the analyst has in mind. One version is known as Leontief's static 'closed' input-output model, and the other as the 'open' input-output model. According to P. N. Mathur (1970, p.2):[1] "The closed input-output model not only provided a framework of meaningful description of economic interconnections, but also gave us the means for finding out the short-period effects of economic policy or data change." This sort of model is used mainly in the short-run when such variables as government expenditures, capital creation, exports, etc., do not experience variations a in this case they can be used as part of he endogenous component of the system.

When, however, the effects of variations of these variables are considered it will be more appropriate to use them as exogenous and make them part of the final demand of the system.[2] In this latter case we have the open version of the input-output model. In other words, if besides the n-industries, the model includes an 'open' sector, which exogenously determines final demand for the product of each of the industries, and which supplies primary inputs not produced by the e-industries, then the model is open.

If the exogenous sector is included in the system as another industry the model will become 'closed'. In this latter case all goods will become intermediate in nature.

When comparing static and dynamic input-output theories Leontief[3] specifies that: "A static theory derives the changes in the variables of a given system from the observed changes in the underlying structural relationship dynamic theory goes further and shows how certain changes in the variables can be explained on the basis of fixed, i.e. invariant structural characteristics of the system... Dynamic Theory thus enables us to derive the

1 Mathur, P. N. (12), p.2.
2 Mathur, PN. (13), p.7
3 Leontief, W. (6), p.53

empirical laws of change of a particular economy from information obtained through the observations of its structural characteristics at one single point of time.”

A. A Brief Mathematical Exposition of Dynamic Models.

When certain considerations are introduced into the static input-output model the system can become dynamic in character, and from the point of view of mathematical analysis the result will be a system of difference or differential equations. One such consideration is capital formation[4]. In static models according to Leontief (1967, p.562-565). “The inputs coefficients do not reflect, however, the stock requirements of the economy ... such an explanation becomes possible as soon as the stock requirements of all the individual sectors of the economy are included in the structural map of the system along with its previously described flow requirements”.[5]

P. N. Mathur defines the Leontieff’s dynamic input-output system as consisting of vector addition of independent Leontief trajectories. The trajectories are defined by the vector of final demand and the rate of growth of the final demand. The dynamic element is introduced by transferring the capital creation part of the model from the final demand to the internal structure of the inputs.[6]

Suppose we want capital creation to be explained by the model (we want it to become endogenous). The necessary procedure will consist of connecting the capital requirement of each sector to the output of that sector by means of some technical coefficient that we call b_{ij} Æ 0. Each of the b_{ij} represents the stock of i-th goods that the industry must have available for each unit of its output. If Sij is the stock of the i-th goods owned by j-th industry and X_j the total output of the j-th industry then:

4 Two more dynamizing considerations are discussed in Chiang, A. C. (2) pp.562-565
5 Leontief, W. (6), pp. 55-56
6 Mathur, P. N., (10) and (9)

(1) $S_{ij} = b_{ij}X_j$

where i = 1, 2, ... n j = 1, 2, ...n

differentiating both sides of equation (1) with respect to time we obtain: I[7].

(2) $S_{ij} = b_{ij}X_j$

The balance equation of the model now becomes:

(3) $X_i = \sum_{j=1}^{n} a_{ij} Xj + \sum_{j=1}^{n} b_{ij} X_j + Y_i$

where **(i = 1, 2,.......n)** **(j = 1, 2,.......,n)**

where X_i is the output currently demanded.

Under equilibrium conditions this output should be equal to the three components in the right-hand side of equation (3). The first component

$$\sum_{j=1}^{n} a_{\boldsymbol{ij}} \boldsymbol{X_j}$$

is equal to the quantity of i-th goods used as intermediate inputs by all industries. The second term, is equal to the quantity of i-th goods demanded for investment purposes and Y_i is the final demand or the quantity of i-th goods used to satisfy final demand. In this case, contour to static models, investment (our second right hand side component) is excluded from the final demand. Equation (3)

$$\sum_{j=1}^{n} b_{ij} X_j$$

represents a system of n-differential equations[8] .

7 . Equation (2) can be interpreted as a continuous version of the 'acceleration principle'.

8 . As in static models the system can also be made 'open' or 'closed' depending on whether the final demand is made endogenous by introducing an additional sector m-th (m=n+1) or in formal term:

$X_i = \sum_{j=1}^{m} a_{ij} X_j + \sum_{j=1}^{m} b_{ij} X$ i=1,2,…m where $a_{im}X + b_{im}X_m = Y_i$

For two industries the balance equation will be as follows:

(4) $X_i = a_{ij}X_i + a_{12}X_2 + b_{11}X + b_{12}X_2 + Y_1$

$X_2 = a_{21}X_1 + a_{22}X_2 + b_{12}X_1 + b_{22}X_2 + Y_2$

In our case the Y's are taken as known.

In the above equations the complete set of capital coefficients of the n-industries (b_{ij}) can be represented as a square matrix of the form:

$$(5) \quad B = \begin{vmatrix} b_{11} & b_{12} & \ldots b_{1n} \\ b_{21} & b_{22} & \ldots b_{2n} \\ . & . & . \\ . & . & . \\ . & . & . \\ b_{n1} & b_{22} & b_{nn} \end{vmatrix}$$

B matrix can be called (as they are by Leontief) the capital coefficient matrix, or the capital structure of the economy.

To solve the system as illustrated by equation (3) the Y's must be known. However, in the case of the closed dynamic input-output system the solution involves only that part which in mathematics is called they system. In this latter case the final demands are irrelevant.

In his article Leontief starts by expressing the solution to the closed or homogeneous system as follows:

$$(6) \qquad X_i(t) = C_1R_{i1}e^{\lambda t} + C_2R_{12}\, e^{\lambda 2t} + C_kR_{ik}e^{\lambda kt} + \ldots + C_nR_{in}\, e^{\lambda nt}$$

In a system of e-equations, each one describes a path; through time of the e-different outputs $X_l(t)$, $X_2(t)$,$X_3(t)$,..., $X_n(T)$. The numerical solutes rewires the determination of the values of the roots of the equations (‘s) the coefficients R_{ik}, the coefficients C’s and the establishment of some initial conditions. The roots may be real or complex, depending on the values of the structural coefficients. However, as Mathur has shown[9], for economic purposes, only one root is relevant and the others will have non-positive vectors associated with them. Therefore, these latter roots have no economic meaning and they can be ignored in economic analysis.

In part 4 of this article Leontief discusses the solution to the ‘original’ open, i.e., non-homogeneous system. The equation for the general solution becomes:

(7) $X_i(t) = C_1R_ie^{\lambda t} + C_2R_{12}\, e^{\lambda 2t} + C_kR_{ik}e^{\lambda kt} + C_mR_{im}\, e^{\lambda mt} + L_i(t)$

However, in view of Mathur’s results (m-l) of these R-vectors will be non-positive and thus non-relevant for economic analysis purposes. Therefore, we can write Leontief’’s results as follows:

(8) $X_i(t) = CK_ie^{\lambda t} + L_i(t)$

In the latter case $L_l(t)$, $L_2(t)$ ... $L_m(t)$ represents the particular part of the general solution (the one including final demand vectors). The shapes of the functions in this case will depend only on the structural coefficients, but also on the shapes taken by the final demand vectors. For instance, final demands can be assumed to be equal to a constant or to grow exponentially. Some writers have given a formulation for the determination of output and investment, when consumption is given, and each constituent of it is subject to exponential growth at different rates[10]. However, this formulation requires the

9 Mathur, P. N., (10) and Mathur, (9), p 76.
10 Stone, R., and Brown, J.A.C., (19).

summation of an infinite series. P.N. Mathur offered an alternative which does not require, or depend on, the “summation of an infinite series and thus opens up the possibility of exalt calculations” of output.[11] P.N. Mathur’s formulation also leads to the determination of an upper limit to the rate of exponential growth for the consumption of any items.[12] (1964 p.73 and part IV p.76)

C. Some Important Applications of Dynamic Models to Economics

Dynamic input-output models have been very practical and useful especially in the field of economic planning at both national and regional levels. They have also been a useful tool in economic projection. When used in conjunction with linear programming models they become even more powerful as planning models because they avoid some of the assumptions involved in dynamic input-output models in their simplest forms. P.N. Mathur has made use of this technique of combining Leontief’s dynamic input-output model with linear programming: “The tools of input-output analysis and of linear programming can be used to trace an efficient path for the transformation of an economy where a new technology appears on the horizon” (1963, pp.39 and 1967, pp.109-130). In another article (Leontief: 1972) he applies dynamic input-output models to regional economic analysis.[13]

A very important application of dynamic input-output models has been in the field of long range projection of economic growth or the use of the dynamic inverse. According to Mathur:[14] “A most important theoretical and empirical advance is Wassily Leontief’s conceptualization and implementation of Dynamic Inverse. That gives us the direct and indirect input requirements generated by the final demand of the year zero for all previous individual years... if delivery to final demand at some specified future date is required, the

11 Mathur, P. N., (9), pp. 74-76 see also his (10).
12 Mathur, P. N., (9) page 73 and part IV page 76.
13 Mathur, P. N., (14), p.39. See also his (15) pp. 109-130.
14 Mathur, P. N., (11)

method determines the ‘productive advances’ necessary today, and at every intervening point of time.... This gives measurable content to the notions of ‘productive advances’ of Quesnay, ‘expanded reproduction’ of Marx, and ‘roundabout production’ of Bohm-Bawerk.”

Leontief uses the dynamic inverse for “long range projection of economic growth”.[15] He also uses the dynamic inverse to describe the direct and indirect input requirements generated by the delivery to final demand of one unit of the products of any of e-industries in some year 0. In addition he made use of the inverse to trace back the direct and indirect effects of a million dollar’s increase in the final demand for the product of certain industry (of the n-number of industries) in year zero on the outputs of other related industries.[16] In relation to the dynamic inverse Mathur made a very important comment:[17] “Another important gain in clarity from looking the dynamic process backward is that spurious mathematical doubts about the stability and convergence of forward Leontief dynamic system are seen to have no validity. The Leontief trajectory having highest rate of growth corresponds to the smallest root of the determining matrix. As the other roots determine a non-positive trajectory, they have no economic relevance.”

In this article we will introduce a model to estimate the dynamic inverse. The next section will be devoted to a brief theoretical exposition of the concept of the dynamic inverse.

2. The Dynamic Inverse

In this section we will follow Leontief’s exposition for illustrative purposes. However, for the empirical implementation of the model a slightly different formulation

15 Mathur, P. N. , (13), p. 13 and (12) pp. 11-12.
16 Leontief, W. (8), pp.17-46.
17 P.N. Mathur, P.N., (13), p.14 and (12).

has been used more akin to his other article on the subject.[18] As already explained elsewhere, each element of the inverse represents the combined direct and indirect inputs required from the row industry to allow for an additional unit of output by the column industry. However, in the dynamic inverse 'such effects' are presented in time series rather than being described by a single number.

A. The Model

Let X represents a column vector of outputs

$X_1^t, X_2^t, \ldots, X_n^t$

F is equal to a column vector of deliveries to final demand (excluding investment of f_1^t, f_2^t, ... , f_n^t).

The structural characteristics of the economy are described by the input-output coefficient matrix (A) and the capital coefficient matrix (B). Both of these are n dimensions, or square matrices. The capital goods produced in year (t) are assumed to be installed in year (t+l).

If more than one (A) and (B) matrices are used then time subscripts will be attached to the matrices and vectors. The interdependence between the outputs of all sectors (let's assume for two successive years) will be represented by:

(9) $X_t - A_tX_t - B_{t+1}(X_{t-1} - X_t) = F_t$

According to Leontief the time subscripts attached to both structural-matrices bring the possibility of using different sets of flow and stock coefficients for different years, thus incorporating technological change into the dynamic system. Equation (9) can be expressed as;

(10) $(I - A_t + B_{t+1})X_t - B_{t+1}X_{t+1}$

If we define $I - A_t + B_{t+1}$ as G_t then equation (10) becomes:

(11) $G_tX_t - B_{t+1}X_{t+1} = F_t$

18 Leontief, W. (7).

In matrix form and for a period of M+1 years we have:

$$
(12)\quad \begin{vmatrix} G_{-m} - B_{-m+1} & & & \\ & G_{-m+1} - B_{-m+2} & & \\ & \cdot & \cdot & \\ & & \cdot & \\ & \cdot & & \cdot \\ & & G_{-2} - B_{-1} & \\ & & G_{-1} - B_{0} & \\ & & & G_{0} \end{vmatrix} \begin{vmatrix} X_{-m} \\ X_{-m+2} \\ \cdot \\ \cdot \\ \cdot \\ X_{-2} \\ X_{-1} \\ X_{0} \end{vmatrix} = \begin{vmatrix} F_{-m} \\ F_{-m+2} \\ \cdot \\ \cdot \\ \cdot \\ F_{-2} \\ F_{-1} \\ F_{0} \end{vmatrix}
$$

The matrix on the left is a triangular matrix with non-zeros entries along and above its main diagonal and all its other elements are equal to zero. As will be explained below all final demands, including F_0, will exclude investment. A solution to the system determines a time series of the e-industry outputs that the economy needs to satisfy the annual final deliveries.

To solve the system represented by equation (12) one needs to solve the last equation, and proceed stepwise by substituting the solution in the next equation until we arrive at the last equation. The solution to the system is given below by equation system (13):

$$
(13)\quad \begin{bmatrix} X_{-m} \\ \cdot \\ \cdot \\ \cdot \\ X_{-2} \\ X_{-1} \\ X_{0} \end{bmatrix} = \begin{bmatrix} G_{-m}^{-1} & \cdots & R_{-m} \cdots R_{-3} R_{-2} G_{-1}^{-1} & R_{-m} \cdots R_{-3} R_{-2} R_{-1} G_{0}^{-1} \\ \cdot & & & \\ \cdot & & & \\ & G_{-2}^{-1} & R_{-2} G_{-1}^{-1} \cdots & R_{-2} R_{-1} G_{0}^{-1} \\ & & G_{-1}^{-1} & R_{-1} G_{0}^{-1} \\ & & & G_{0}^{-1} \end{bmatrix} \begin{bmatrix} F_{-m} \\ \cdot \\ \cdot \\ \cdot \\ F_{-2} \\ F_{-1} \\ F_{0} \end{bmatrix}
$$

Where $R_t = G_t^{-1} B_{t-1} = (I - A_t + B_{t+1})^{-1} B_{t-1}$

In the system represented by equations (13) every element of the square matrix on the right is itself a matrix. For instance, the column in the square matrix with the G_0^{-1} represents the direct and indirect inputs required to deliver a unit of final demand by the n-industries in the year zero.

According to Leontief[19]: “Any analytical description of a dynamic process extending beyond the two limits has to be sealed up against “unraveling” at one of its two ends... That involves making some arbitrary but nevertheless quite specific factual assumptions concerning the course that the development of the economy in question will take beyond the terminal year of the projection”.

In this article, used for projection purposes, he sealed the system at the forward end by assuming the G_0^{-1} is equal to:

(14) $G_0^{1} \equiv [I - A + (i - \alpha)B_0]^{-1}$ instead of $[I - A - B_0]^{-1}$

This procedure allows final demand in the base year (F_0) to be defined as a concept of final demand excluding investment. In our case, we have assumed a slightly different equation for the base year.

The system as represented by equation (12) is designed for cases where more than one capital coefficient and input-output matrices are available. In our case only one input-output coefficient and capital coefficient matrix will be used, since no other alternative is available, due to data limitations.

In Leontief’s model as represented by equation (13) G_0^{l} represents the input requirements that must be filled in the base year zero, $R_{-1}\, G_0^{-1}$ specifies the requirements one year bettered and

$R_{-m}\ldots R_{-2}R_{-1}G_0^{-1}$

represents the inputs that have to be provided m-years before the final deliveries are made in the base year.

In our case the time subscripts were eliminated from the structural constants. According to Leontief, in the absence of technical change, the elements of each column can be described in receding order by the same simple geometric series.

(15) $G^{-1}, RG^{-1}, R^2G^{-1}, \ldots, R^tG^{-1}, \ldots, R^mG^{-1}$

3. The Dynamic Inverse: A Model for Empirical Estimates

The dynamic inverse can be estimated by choosing a base year let's say calendar year 1984, and working the process backwards to calendar year, let's say 1954. In what follows we present the mathematical equations and symbols suggested for empirical computations[19].

Definitions:

. B = Capital Coefficient Matrix (32 by 32)

A = Input-Output Coefficient Matrix (32 by 32)

F = Column Vector of final demands, excluding investment

X = Column Vector of output

X_0 = output in the base year of 1984

α = $(X_{t+1} - X_t)/ X_t$ (assumed same for each commodity)

Go= $(I-A-\propto B)$ equal to bane year matrix (1984)

G_1 = $(I-A+B)$ equal to matrix used from 1983 backwards to 1954

$Y_0 = (I-A-\_B)^{-1}F_o$ (where the subscript 0 means base year 1984) or $Y_0 = G_0^{-1}F_0$

$X_0 = Y_0$

. Y_{-t} = $(I-A+B)^{-1} F_{-t}$ t = 1983, 1982, 1981, ... 1954

(for t = -1, -2, -3, ..., -30)

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19 . According to P.N. Mathur one important gain in clarity by looking at the process backwards is to show that mathematical doubts concerning the stability and covergence of the forward, process are seen to have no validity.

$D = (I-A+B)^{-1} B$

The Equations:

For the first two years (1983 and 1984) the equations of the solution of the system will look like:

For the whole period of 30 years, plus the base year of 1984, the solution to the system will be given by the following system of equations shown in matrix notation.

(16) $$X^0 = (I-A-\alpha B)^{-1}F_0 = Y_0$$

(17) $$X_{-1} = (I - A + B)^{-1}F_{-1} + \{|(I - A + B)^{-1}|\ |(I - a - \alpha B)^{-1}|\}$$

(18) $$\begin{bmatrix} X_0 = G_0^{-1}F_0 = Y_0 \\ X_{-1} = Y_{-1} + DX_0 \\ X_{-2} = Y_{-2} + DX_{-1} \\ X_{-3} = Y_{-3} + DX_{-2} \\ . \quad . \quad . \\ . \quad . \quad . \\ . \quad . \quad . \\ X_{-30} = Y_{-30} + DX_{-29} \end{bmatrix}$$

Concluding Comments

In this short paper I have shown a model to estimate a dynamic inverse empirically. It should be born in mind that the data requirement for these purposes is enormous..It needs for instance, a capital-coefficient matrix (at present is under estimation by this author) which is quite difficult to obtain.

The model for the computation of the dynamic inverse (or we may say the solution of a system using the dynamic inverse) has not not been presented here as an ambitious project. As Leontief expresses it.[20]: “the dynamic input-output system .... can

20 Leontief, W. (8) page 30.

be of little help in the derivation of the Golden Rules of economic growth ... the dynamic inverse is primarily a storehouse of systematically organized factual information".

As shown here it can be used as an experiment in economic history. As P.N. Mathur expresses the dynamic inverse has "become an eminently suitable tool ... for rewriting economic history as a unified but disaggregate structural movement of economic activities".[21]

21 Mathur, P. N. (13), page 14 and also (12), page 12.

References

Allen, R.G.D., (1959). ***Mathematical Economics***, second edition, (McMillan and Co. Ltd., London).

Chiang, A.C., (1967). ***Fundamental Methods of Mathematical Economics***, (McGraw-Hill Book Co., New York).

Dorfman, R.P., Samuelson, P. and Solow, R., (1957). ***Linear Programming and Economic Analysis***, (McGraw-Hill Book Co., New York,).

Gandolfo, G., (1971). ***Mathematical Method and Models in Economic Dynamics***, (North Holland Publishing Company Amsterdam).

Hawkins, D. and Simon, H.A. (1949). "Note: Some Conditions of Macro-Economic Stability", ***Econometrica***, 17.

Leontief, W., (1953). "Dynamic Analysis" in ***Studies in the Structure of the American Economy***, (Oxford University Press, London).

----(1967). "An Open Dynamic System for Long Range Projection of Economic Growth" in P.N. Mathur and R. Bharadwaj, ***Economic Analysis in Input-Output Framework***, Vol. 1.

---(1967). **"The Dynamic Inverse"** in A.P. Carter and A. Brody. ***Contributions to Input-Output And Framework to Input-Output Analysis***, Vol. l.

Mathur, P. N., (l964). "Output and Investment for Exponential Growth in Consumption: An Alternative Formulation: and Derivation of Their Technological Upper Limits", Review of Economic Studies, Vol. 31.

---(1965). "A Modified Leontief Dynamic Model and Related Price System", Indian Economic Journal, Number 3.

---(1972). **"Multi-regional Analysis in Dynamic Input-Output Framework"** in A. Brody and A.P. Carter (eds), ***Input-Output Techniques***, (North Holland Publishing Co., Amsterdam).

---(1970). **"Introduction"** to A.P. Carter and A. Brody, **Contribution to Input-Output Analysis**, Vol. I (North Holland Publishing Co., Amsterdam).

Marthur, P. N. (1969). "Input-Output Framework for Explorations in Theoretical and Empirical Economic Research" in P.N. Mathur and P. Venkatramaih (eds.), E***conomic Analysis in Input-Output Framework with Indian Empirical Explorations***, Vol. 2.

, (1963). "An Efficient Path for the Technologicaconomy" in T. Barna (ed.), ***Structural Interdependence and Economic Development***, McMillan & Co. Ltd., London.

, (Poono, India, 1967). “An Application of Dynamic Input-Output Model for Planning” in P.N. Mathur and R. Bharadwaj (eds.), ***Economic Analysis in Input-Output Framework***, Vol. 1.

Morishima, Michio, (1958). “Prices Interest and Profits in a Dynamic Leontief System”, ***Econométrica***, Vol. 26.

Ruíz, Anget Luis, (1976). “Capital Imports and the Economic Development of Puerto Rico:Studies in Input-Output Framework”, Unpoblished Ph.D. Dissertation, University of Wales, G.B.

Solow, R. M., (1959). “Competitive Valuation in Dynamic Input-Output System” ***Econométrica***, 27.

Stone, R. and J.A.C. Brown, (June, 1962). “Output and Investment for Exponential Growth in Consumption”, ***Review of Economic Studies***.

Weiskoff, R., A Multisectoral Model of Employment, Growth and Income Distribution in Puerto Rico, (Puerto Rico Planning Board and Yale University).